

## IN SEARCH OF CLASSICAL TRAJECTORIES \*

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## INTRODUCTION

In this short talk, I will emphasize some of the key points of a new Minkowski space formulation of nonperturbative contributions to quantum scattering [1], described at length by Hsu elsewhere in this volume. I will especially comment on some of the practical considerations of computing classical trajectories.

Our formulation [1] is based on a stationary phase approximation to a scattering amplitude involving an initial two-particle state. A real or complex classical stationary trajectory is determined by an initial value or boundary value problem, respectively. The work establishes the role of Minkowski trajectories for quantum scattering amplitudes in weakly coupled field theories, and outlines a procedure for finding such trajectories, in principle via computation. Phenomena of particular interest in this regard involve final states with (i) large numbers of particles ( $N \sim 1/g^2$  where  $g$  is the coupling constant of the theory), and/or (ii) anomalous violation of global quantum numbers, like fermion number violation in the electroweak theory.

Standard perturbative methods cannot describe these phenomenon. Current non-perturbative Euclidean space methods break down at high energies ( $E \sim M_w/g^2$  in electroweak theory) because they do not account for the non-vacuum boundary conditions appropriate for a scattering problem. Our approach is to find the stationary point of a scattering amplitude, to include the effects of initial and final states on the stationary trajectory, and then to assess in what regime the expansion around this classical trajectory is controlled.

The formalism was illustrated for the simple case of a real scalar field [1]. A distinction is then made between real and complex stationary trajectories of the scattering amplitude of the real scalar field. I emphasize this here because the computation of the two types of trajectories could in practice be quite different.

## REAL TRAJECTORIES

A given initial state on a time slice  $t = T_i$  implies initial values for  $\phi$  and  $\dot{\phi}$ , which are sufficient to determine a real trajectory uniquely [1]. The initial condition expressed in terms of Fourier components of the field is

$$\phi_i(\vec{k}) = \frac{1}{\sqrt{2}\omega_{\mathbf{k}}} \left( u_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}T_i} + u_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}}T_i} \right), \quad (1)$$

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where the negative frequency component is

$$u_{\mathbf{k}} = \frac{\alpha_R(\vec{k}) + \alpha_L(\vec{k})}{\left(1 + \int d^3k \alpha_R(\vec{k}) \alpha_L(\vec{k})\right)^{1/2}}, \quad (2)$$

with the (normalized) right- and left-moving wave packets  $\alpha_R(\vec{k})$  and  $\alpha_L(\vec{k})$  of the two-particle initial state. One assumes here that the field behaves freely for times  $t < T_i$ , so that the field and its time derivative are related in the obvious way. The field evolves forward in time from (1) according to the source-free equations of motion

$$\frac{\delta S}{\delta \phi(x)} = 0. \quad (3)$$

So, a real trajectory is determined by an initial value problem. A real trajectory connects every initial wave packet state with a unique final state. The corresponding scattering amplitude is then not exponentially suppressed, since its action is real. But, the final state may or may not be interesting from the point of view of (i) and (ii) above. So, one must search the space of fields for interesting trajectories, by varying the initial wave packets. One might hope to find evidence of instabilities for the generation of long wavelength modes from the initial short wavelength modes, the signal of many (soft) particles in the final state.

I stress that a negative result to the search does not rule out the existence of unsuppressed amplitudes involving interesting final states. There is no one-to-one correspondence between real trajectories and scattering amplitudes. (more on this below) There may well be amplitudes not dominated in any way by real classical trajectories !

## Hints for Classical Trajectories

Several previous investigations into the classical behavior of a variety of field theories provide some hints about the behavior of their real trajectories. I refer you to the discussion and references contained in Gould et al. [1] Here I would like to mention one most promising situation: evidence for an instability in Yang-Mills theory. Müller et al. [2] have considered the classical stability of a stationary mono-color wave in Yang-Mills theory (in  $A_0^c = 0$  gauge):

$$A_i^c(x, t) = \delta_{i3} \delta_{c3} A \cos k_0 x \cos \omega_0 t, \quad (4)$$

where  $c$  is a color and  $i$  is a spatial index. Small amplitude variations of the field in directions of different ( $c \neq 3$ ) color are found to lead to an instability with long wavelength. This implies the existence of a classical trajectory connecting initial high energy plane waves to long wavelength modes in the final state. It therefore suggests that the corresponding  $2 \rightarrow \text{many}$  gluon scattering amplitude may be unsuppressed !

For the purpose of the scattering problem, it would be interesting to see whether the instability in Yang-Mills theory persists for wave packets of finite spatial extent. As the width of a wave packet in momentum space decreases, approaching the plane wave limit (4) in which the instability has been observed, its amplitude decreases also, with the normalization held fixed [1]. However, since the instability persists for arbitrarily small amplitude [2], one may hope that it still produces many long wavelength modes.

In the context of electroweak theory, it will also be necessary to consider how the instability is modified in the presence of an explicit symmetry breaking scale. Some benchmark figures for a study of electroweak theory might be the following. One could consider initial wave packets with average momenta

$$k_{avg} \sim M_w / g^2,$$

and width

$$\Delta k \sim M_w .$$

Such wave packets overlap for a very short time for weak coupling,  $\sim g^2/M_w$ , during which an instability must generate long wavelength amplitudes. I emphasize though that a quantitative understanding of these estimates should be obtainable from direct computation of classical trajectories in a chosen field theory.

## COMPLEX TRAJECTORIES

Complex trajectories of the real field arise from an analytic continuation of the path integral describing the scattering amplitude [1]. Complex trajectories can in principle connect a given initial state with *any* final state, due to the additional degrees of freedom. They include the case of classically forbidden transitions, in the case where initial and final states are based on different vacua separated by an energy barrier in configuration space. This is the situation relevant for processes with anomalous fermion number violation in electroweak theory and center of mass energies below the sphaleron energy,  $\sim M_w/g^2$ . Complex trajectories will have complex actions, so the corresponding amplitudes can in general have some exponential suppression.

The field is initially free, so it has a form similar to (1)

$$\phi_i(\vec{p}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left( u_{\mathbf{p}} e^{-i\omega_{\mathbf{p}}T_i} + v_{\mathbf{p}} e^{i\omega_{\mathbf{p}}T_i} \right) , \quad (5)$$

but where  $u_{\mathbf{p}}$  and  $v_{\mathbf{p}}$  are now independent complex functions. The initial boundary condition is satisfied by

$$u_{\mathbf{p}} = \frac{\alpha_R(\vec{p})}{\int d^3k \alpha_R(\vec{k}) v_{\mathbf{k}}} + \frac{\alpha_L(\vec{p})}{\int d^3k \alpha_L(\vec{k}) v_{\mathbf{k}}} , \quad (6)$$

where  $v_{\mathbf{k}}$  is arbitrary. The initial configurations which satisfy (5), (6) have negative frequency modes which “resemble” the wave packet  $\alpha_L + \alpha_R$  (up to complex multiplicative factors) and positive frequency modes which are arbitrary.

An additional degree of freedom is present, which must be determined by the additional information in the final state. The final state enforces a condition on the positive frequency part of the complex trajectory at final time  $T_f$ . The negative frequency part of the field remains independent and undetermined. The boundary condition takes the form

$$\phi_f(\vec{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( c_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}T_f} + b_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}}T_f} \right) , \quad (7)$$

for a final state  $|\mathbf{b}^*\rangle$ . Finally, the field evolves again with the source-free equations of motion, (3).

So, a complex trajectory is determined by a boundary value problem, with boundary conditions given by the initial and final states. One can consider solving this problem with a search procedure similar to the case of a real trajectory. A given  $u_{\mathbf{p}}$  specified by the initial state gives rise to any number of final states, by varying  $v_{\mathbf{k}}$ . Alternatively, a direct solution of the boundary value problem for given initial and final states may be possible.

## A WINDOW OF INTEREST

Once an interesting classical trajectory is found, one must turn to the problem of the quantum corrections to determine the regime in which the semiclassical expansion

is controlled. I would like to emphasize in this regard how our semiclassical expansion differs subtly from the more familiar instanton expansion. In the familiar semiclassical expansion around an instanton, the vacuum boundary conditions of the classical field allow us to scale the field by the coupling constant, both in the boundary conditions and in the action

$$S[\phi] = \frac{1}{g^2} \hat{S}[g\phi]$$

so that  $\hat{S}$  is independent of the coupling  $g$ . Then, the expansion is good, and the instanton contribution dominates, for sufficiently small coupling  $g$ , where the semiclassical exponent is large.

In our formulation, we account for the non-vacuum boundary conditions relevant to a scattering amplitude. As such, the boundary conditions are fixed by physical initial and/or final states, and the scaling above cannot be made. Now, we must require the coupling to be sufficiently large to produce an instability for the production of long wavelength modes from short wavelength modes. Meanwhile, our experience with ordinary perturbation theory and the semiclassical expansion leads us to expect that the coupling must be sufficiently small to control the expansion. Thus, there must be a *window* in the coupling in order that our method be both interesting and controlled. While we have not proven that a window exists in any field theory, the means by which to do so are clear. The upper bound on the coupling can be determined by computations of classical solutions to the field equations, of the type outlined above. The lower bound on the coupling will have to come from computations of the quantum corrections, as discussed in Gould et al. [1]

## COMMENTS

Current methods are inadequate to describe nonperturbative contributions to scattering amplitudes at very high energies. An outstanding example is the unsolved problem of the rate of fermion number violation in high energy collisions ( $E \sim M_w/g^2$ ) in electroweak theory. I will close by emphasizing that *any* solution of the classical problems which I have outlined here corresponds to the stationary point of *some* scattering amplitude, and therefore yields *some* nontrivial information about the nonlinear aspects of quantum field theory, entirely inaccessible to perturbation theory. While much is known about semiclassical calculations in the vacuum sectors of field theories, our work points towards the need for a study of a much wider class of classical solutions, involving multi-particle boundary conditions.

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## References

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